THE INTEGRAL ROOTS OF CERTAIN INEQUALITIES

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1. Introductory. Let the residue, mod 1, of any number x be denoted by $F(x)$. Let d be any number and β, β' be positive numbers less than 1.

The following paper is concerned with integral solutions, Y, of the inequalities

$$\beta > F(Yd) > \beta'. \tag{1}$$

If d is a commensurable number, P/Q, P and Q being positive integers prime to each other, and all positive integral roots $[Y_r]_{r=1}^{r=k}$ less than Q have been found by trial, the complete solution is given by

$$Y = Y_1,\ Y_2,\ \cdots\ Y_k, \text{ mod } Q. \tag{2}$$

That is, the series of roots of (1), arranged in order of magnitude possesses a period Q.

If d is not a commensurable number, this period disappears but is replaced by a quasi-periodicity, the regularity of which is marred by gaps at certain points to be found later.

Further, even when d is commensurable, this quasi-periodic structure may be found within the regular period Q. This is illustrated by the following example:

Let
$$\tfrac{3}{4} > F(Y\tfrac{7}{58}) > \tfrac{1}{4}. \tag{3}$$

The solutions $Y_1,\ Y_2,\ \cdots\ Y_k$ are as follows:

3, 4, 5, 6,				3,	4,	5,	6
11, 12, 13, 14,		8 +		3,	4,	5,	6
19, 20, 21, 22,		2·8 +		3,	4,	5,	6
27, 28, 29, 30, 31	25		+ 2,	3,	4,	5,	6
36, 37, 38, 39,	25 +	8 +		3,	4,	5,	6
44, 45, 46, 47,	25 +	2·8 +		3,	4,	5,	6
52, 53, 54, 55.	2·25		+ 2,	3,	4,	5.	

The second method of arrangement shows clearly that within the regular period of $58 (= 2\cdot 25 + 8)$ there is a clearly marked quasi-periodicity, like that with which the recurrence of eclipses has made us familiar. It seems

hardly correct to replace the term quasi-periodicity by the shorter word periodicity because neither the period of repetition nor the group repeated is quite permanent.

There is a primary group, 3, 4, 5, 6, to which the term 2 may be added, or from which the term 6 may be omitted, and this we may denote by S_1.

There is a secondary group which we will call S_2, which consists of the terms

$$S_1, \qquad S_1 + 8, \qquad S_1 + 2\cdot 8,$$

where it is to be understood that the final number is to be added to each member of the group S_1.

The final group S, which is, in this case of the third order, consists of the terms

$$S_2, \qquad S_2 + 25, \qquad S_1 + 2\cdot 25.$$

This group is repeated with perfect regularity.

It is the object of the present paper to develop a method by which this periodic structure can be studied in detail. Actually to exhibit this structure, built up of one period within another, by a system of formulas which includes all possible cases is too complicated a result to be reproduced here.

It is of interest to note that the results are not limited to *commensurable* values of d, β or β'.

The solution of the present problem was attempted in order to answer questions raised by the paper on shadow rails which immediately follows it.

The writer has found no references pertinent to the subject.

2. Notation. As may be readily guessed, it is the expansion of d as a simple continued fraction which is the initial step from which all else follows.

Suppose d to be expanded in a simple continued fraction as below.

Let
$$d = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots\frac{1}{a_n + e_n}, \tag{4}$$

where the a's are positive integers, $0 < e_n < 1$,

and
$$e_{n-1} = \frac{1}{a_n + e_n}. \tag{5}$$

The coefficients a_r are most readily determined by applying the ordinary process for finding the highest common factor of two numbers to the numbers d and 1. In the case where d is a commensurable number, P/Q, it is more

convenient to apply the process to P and Q. The equations thus obtained will be the same as those which follow, multiplied throughout by Q.

Thus let

$$
\begin{aligned}
d &= a_1 \cdot 1 + w_1, & 0 < w_1 < 1, \\
1 = w_0 &= a_2 \cdot w_1 + w_2, & 0 < w_2 < w_1, \\
w_1 &= a_3 \cdot w_2 + w_3, & 0 < w_3 < w_2, \\
&\cdots\cdots\cdots \\
w_{n-2} &= a_n \; w_{n-1} + w_n, & 0 < w_n < w_{n-1}.
\end{aligned} \tag{6}
$$

The quantities w_r, not considered in the usual treatment of continued fractions, are fundamental in the present discussion.*

Let the nth convergent be denoted by p_n/q_n. Since this is obtained from the $(n-1)$th convergent by the substitution of $a_{n-1} + 1/a_n$ for a_{n-1}, it is easily seen that the following equations are true:

$$
\begin{aligned}
p_1 &= a_1 \cdot 1 + 0, & q_1 &= a_1 \cdot 0 + 1, \\
p_2 &= a_2 \cdot p_1 + p, & q_2 &= a_2 \cdot q_1 + 0, \\
p_3 &= a_3 \cdot p_2 + p_1, & q_3 &= a_3 \cdot q_2 + q_1, \\
&\cdots\cdots & &\cdots\cdots \\
p_n &= a_n \cdot p_{n-1} + p_{n-2}, & q_n &= a_n \cdot q_{n-1} + q_{n-2}.
\end{aligned} \tag{7}
$$

Further, equations (6) may be put into the same form as those just written:

$$
\begin{aligned}
-w_1 &= a_1 \cdot 1 - d, \\
w_2 &= a_2 \cdot (-w_1) + w_0, \\
-w_3 &= a_3 \cdot w_2 - w_1, \\
&\cdots\cdots\cdots \\
(-1)^n w_n &= a_n \cdot (-1)^{n-1} w_{n-1} + (-1)^{n-2} w_{n-2}.
\end{aligned} \tag{8}
$$

Lastly, if in equations (7) we multiply each of the equations involving the convergents q_r by $-d$ and add to the corresponding equations involving p_r, we find by comparison with equations (8) that

$$(-1)^n w_n = p_n - dq_n,$$

or

$$q_n d = p_n + (-1)^{n+1} w_n. \tag{9}$$

* It follows by comparison with equations (4), (5) that

$$e_1 = w_1, \quad e_2 = w_2/w_1, \; \ldots \; e_n = w_n/w_{n-1}, \quad \text{whence} \quad w_n = e_1 e_2 e_3 \ldots e_n.$$

The mode of construction of equations (6) ensures that the quantities w_r form a sequence of positive quantities approaching zero as a limit. Equation (9) shows that they are fundamental in calculating the residues, mod 1, of integral multiples of d.

It will be shown (Theorem B) that any positive quantity β, less than 1, can be expressed as the sum of either a finite series or a convergent infinite series of integral multiples of the quantities w_r. If in this series $(-1)^{n+1} w_n$ is replaced by q_n, we obtain a corresponding series, divergent if infinite, such that if B_n denote the sum of its first n terms,

$$\lim_{n=\infty} F(B_n d) = \beta.$$

The converse process of finding $F(Bd)$, when B is given, will in general best be accomplished directly by multiplication rather than indirectly by first expanding B in a series of multiples of the convergents q_r.

3. The Fundamental Theorems. Let $b_1, b_2, \cdots b_n$ be positive integers and let

$$B_n = \sum_{r=1}^{n} b_r q_r, \tag{10}$$

or, what comes to the same thing, let

$$B_n = b_n q_n + B_{n-1}. \tag{11}$$

THEOREM A. *Any positive integer can be uniquely expressed as a series* (10) *by means of the inequalities*

$$q_{n+1} > B_n \geqq 0. \tag{12}$$

Further, the coefficients, b_n so determined satisfy the inequalities

$$a_{n+2} \geqq b_{n+1} \geqq 0, \quad when \quad b_n = 0, \qquad n \neq 0,$$
$$and \quad a_{n+2} - 1 \geqq b_{n+1} \geqq 0, \quad when \quad n = 0 \quad or \quad b_n \neq 0, \quad n \neq 0. \tag{14}$$

If inequalities (12) hold good for B_n, it follows from equations (7) that two cases arise in which either

$$\text{(i)} \qquad q_{n+1} > B_n \geqq a_{n+1} q_n, \tag{15}$$

$$\text{or} \qquad \text{(ii)} \qquad a_{n+1} q_n > B_n \geqq 0. \tag{16}$$

In each case assume that inequalities (12) hold good for B_{n-1}. It then follows from (11) that

$$q_n > B_n - b_n q_n \geqq 0. \tag{17}$$

That is, b_n is the greatest integer contained in B_n/q_n, and the coefficients b_n are uniquely determined.

In case (i), therefore, inequalities (15) and (17) give

$$q_{n+1} + q_n > (b_n + 1)q_n > a_{n+1}q_n.$$

Hence, from (7)

$$a_{n+1}q_n + q_{n-1} > b_n q_n > (a_{n+1} - 1)q_n,$$

that is
$$b_n = a_{n+1}. \tag{18}$$

Further, if inequalities (12) hold for B_{n-2}, we see from (17) that

$$q_{n-1} > B_{n-1} - b_{n-1}q_{n-1} \geqq 0,$$

and from a second application of (11),

$$B_n - a_{n+1}q_n - b_{n-1}q_{n-1} \geqq 0.$$

From (7) and (15)
$$(1 - b_{n-1})q_{n-1} > 0,$$

whence
$$b_{n-1} = 0. \tag{19}$$

In case (ii), it follows similarly from (16) and (17) that

$$a_{n+1}\, q_n > b_n q_n > -q_n,$$

whence
$$a_{n+1} - 1 \geqq b_n \geqq 0. \tag{20}$$

Equations (18), (19), (20) show that the coefficients b_n satisfy inequalities (14) when $n \neq 0$. It follows directly from (7) and (12) that, also when $n = 0$, inequalities (14) are satisfied.

Corollary. Conversely, if inequalities (14) are satisfied it follows that inequalities (12) are satisfied and hence that there is only one expansion of B_n in which the coefficients satisfy (14).

The proof is as follows. Assume that

$$q_{n+1} > B_n \geqq 0.$$

If
$$(a_{n+2} - 1) \geqq b_{n+1} \geqq 0,$$

$$(a_{n+2} - 1)\, q_{n+1} \geqq b_{n+1}q_{n+1} \geqq 0,$$

and therefore
$$q_{n+2} - q_{n+1} \geqq B_{n+1} \geqq 0, \qquad \text{from (7) and (11).}$$

If
$$b_{n+1} = 0, \quad b_{n+2} = a_{n+3},$$

from (11) $$B_{n+2} = a_{n+3}q_{n+2} + B_n,$$

and therefore $$a_{n+3}q_{n+2} + q_{n+1} > B_{n+2} > 0,$$

that is $$q_{n+3} > B_{n+2} > 0.$$

That is, if inequalities (12) are true for n, they are true for $n+1$. But from (7) and (14) it follows that they are true when $n = 1$. They are therefore satisfied for all positive integral values of n, which proves the theorem.

It follows from equations (9) and (10) that

$$B_n d = \sum_{r=1}^{n} b_r p_r + \sum_{r=1}^{n} (-1)^{r+1} b_r w_r. \tag{21}$$

That is, the residue, mod 1, of $B_n d$ can be uniquely expressed as a series of multiples of the quantities w_r, and the coefficients b_r, satisfy (14). This raises the converse question, can any positive quantity β, less than unity be expressed as the sum of such a series, and so determine a multiplier B such that

$$F(Bd) = \beta.$$

First it is necessary to consider the nature of the convergence of such a series.

Let $$\beta_n = \sum_{r=1}^{n} (-1)^{r+1} b_r w_r. \tag{22}$$

Let the first odd and even coefficients which are not zero be b_{2u-1} and b_{2s} respectively.

Equations (6) enable us to write

$$w_n > a_{n+2} w_{n+1}. \tag{23}$$

It is convenient to note the following consequences of inequalities (14) and (23), when use is made of equations (6).

(i) If $s \geqq u = 1$, $$1 - w_1 > \beta_n > w_2,$$

and if $s \geqq u \geqq 2$, $$w_{2u-2} > \beta_n > w_{2u}. \tag{24}$$

(ii) If $u \geqq s + 1$, $$-w_{2s+1} > \beta_n > -w_{2s-1}. \tag{25}$$

(iii) If $n > r$, $b_{r+1} \neq a_{r+2}$,

$$w_r - w_{r+1} > (-1)^r(\beta_n - \beta_r) > -w_{r+1}. \tag{26}$$

(iv) If $n > r$, $b_{r+1} = a_{r+2}$,

$$w_r > (-1)^r(\beta_n - \beta_r) > w_r - w_{r+1}. \tag{27}$$

(v) In all cases, $1 - w_1 > \beta_n > -w_1$.

The above inequalities are written out for the case in which d is incommensurable. When d is commensurable a sign of equality must be inserted either at the upper or lower limit. The limit at which it must be placed depends on whether the last a_n which is not zero has an odd or even suffix.

It is clear that the series β_n is convergent for all possible values of the coefficients b_r, lying within the limits prescribed by (14).

Of the above results, inequalities (26), on account of their generality, form the best starting point for the expansion of any positive number less than 1 in a series β_n.

Let

$$\beta - \beta_0 = \beta_n + (-1)^n \rho_n, \tag{28}$$

where $\beta_0 = 0$ or 1 according as $\beta <$ or $\geqq 1 - w_1$. (29)

It follows from (22) and (28) that

$$\rho_n + \rho_{n+1} = b_{n+1} w_{n+1}. \tag{30}$$

Inequalities (26) suggest the following as a means of determining the expansion (28):

$$w_n > \rho_n + w_{n+1} \geqq 0. \tag{31}$$

THEOREM B. *Any positive quantity β, less than* 1, *can be expanded uniquely in the series* (28), *if each positive integer b_n is determined for successive values of n, when possible, from the inequality for ρ_n in* (31) *and is otherwise zero. Further, the values of b_n so obtained satisfy* (14).

Making use of equations (30), we may write (31) in the form

$$w_n > b_n w_n - \rho_{n-1} + w_{n+1} \geqq 0. \tag{32}$$

These inequalities determine b_n as the least integer not less than

$$(\rho_{n-1} - w_{n+1}) / w_n.$$

Again, from (31), we assume

$$w_{n-1} > \rho_{n-1} + w_n \geqq 0,$$

whence, from (6),

$$(a_{n+1} - 1) w_n > \rho_{n-1} - w_{n+1} \geqq -w_n - w_{n+1},$$

and, therefore

$$a_{n+1} - 1 \geqq b_n \geqq -1. \tag{33}$$

If

$$w_{n+1} \geqq \rho_{n-1} + w_n \geqq 0, \tag{34}$$

b_n as determined by (32) would be negative, -1 in fact; we write $b_n = 0$ and in this case we find by a double application of equations (30) to inequalities (31) for ρ_{n+1} that

$$w_{n+1} > b_{n+1} w_{n+1} + \rho_{n-1} + w_{n+2} \geqq 0,$$

which determines b_{n+1} as the least integer not less than $(-\rho_{n-1} - w_{n+2})/w_{n+1}$.

Further, from (34),

$$2w_{n+1} > (b_{n+1} + 1)w_{n+1} - w_n + w_{n+2} \geqq 0.$$

Comparison with equations (6) shows that

$$2w_{n+1} > (b_{n+1} + 1 - a_{n+2})w_{n+1} \geqq 0,$$

that is

$$b_n = 0, \quad b_{n+1} = a_{n+2} \quad \text{or} \quad a_{n+2} - 1. \tag{35}$$

Hence if one value of ρ_n is found satisfying inequalities (31), the series is uniquely determined for any number of terms greater than n.

But when $\beta < 1 - w_1$, $\rho_0 = \beta$, and we have

$$w_0 > \rho_0 + w_1 > 0.$$

And when $\beta \geqq 1 - w_1$, $\rho_0 = \beta - 1$, and again

$$w_0 > \rho_0 + w_1 \geqq 0.$$

That is, the series is in all cases uniquely determined.

Lastly, it is clear from inequalities (33), (35) that the coefficients b_n, satisfy inequalities (14).

Corollary. If ρ_n does not satisfy inequalities (31), it follows from (34) by the means of (30) that, since $b_n = 0$,

$$w_n + w_{n+1} \geqq \rho_n + w_{n+1} \geqq w_n. \tag{36}$$

4. Applications. As a half-way step towards finding the integral roots of inequalities (1), it is convenient to formulate the conditions for

$$F(Yd) \gtrless \beta,$$

where β has the value assigned in (28).

Let Y be expanded by Theorem A into the form

$$Y = Y_n + S_n, \tag{37}$$

where
$$Y_n = \sum_{r=l}^{n} y_r q_r, \quad S_n = \sum_{r=n}^{n+m} y_r q_r. \tag{38}$$

It follows from equations (9), that if
$$\eta_n = \sum_{r=l}^{n} (-1)^{r+1} y_r w_r, \quad (-1)^n \sigma_n = \sum_{r=n}^{n+m} (-1)^{r+1} y_r w_r, \tag{39}$$

then
$$F(Yd) = \eta_0 + \eta_n + (-1)^n \sigma_n, \tag{40}$$

where η_0 is 0 or 1 as l is odd or even, as may be seen by referring to inequalities (24), (25).

It follows from equations (6) that starting out from the critical value $1 - w_1$, the interval from 0 to 1 is completely made up of the intervals separating

$$1 - w_1,\ w_2,\ w_4,\ w_6 \cdots \text{ on the one hand, and}$$
$$1 - w_1,\ 1 - w_3,\ 1 - w_5,\ \cdots \text{ on the other.}$$

Two cases are best considered separately, in which either

$$\text{I,} \quad \beta < 1 - w_1, \qquad \text{or} \qquad \text{II,} \quad \beta \geqq 1 - w_1.$$

I. In the first case, let β be contained in the interval $w_{2u-2} w_{2u}$. From inequalities (24) (25)

$$\begin{aligned} &F(Yd) > \beta, \quad \text{if} \quad l = 2k - 1,\ k < u, \quad \text{or if} \quad l = 2s, \\ &F(Yd) < \beta, \quad \text{if} \quad l = 2k - 1,\ k > u. \end{aligned} \tag{41}$$

If $l = 2u - 1$, write $\beta = b_{2u-1} w_{2u-1} - \rho_{2u-1}$, where, from (31),
$$w_{2u-1} > \rho_{2u-1} + w_{2u} \geqq 0. \tag{42}$$
$$F(Yd) = y_{2u-1} w_{2u-1} - \sigma_{2u-1},$$

where, from (26) and (27),
$$w_{2u-1} > \sigma_{2u-1} + w_{2u} > 0. \tag{43}$$

It follows from (42) and (43) that
$$w_{2u-1} > |\rho_{2u-1} - \sigma_{2u-1}| \geqq 0,$$

whence, if $l = 2u - 1$, $F(Yd) \gtrless \beta$ according as $y_{2u-1} \gtrless b_{2u-1}$. (44)

$$\text{If } [y_r = b_r]_{r=2u-1}^{r},\ y_{r+1} \neq b_{r+1},$$
$$\beta = \beta_r + (-1)^r (b_{r+1} w_{r+1} - \rho_{r+1}),$$
$$F(Yd) = \beta_r + (-1)^r (y_{r+1} w_{r+1} - \sigma_{r+1}).$$

Three cases now arise, in which

(i) $$w_{r+1} > \rho_{r+1} + w_{r+2} \geqq 0,$$
$$w_{r+1} > \sigma_{r+1} + w_{r+2} > 0,$$
whence $$w_{r+1} > |\rho_{r+1} - \sigma_{r+1}| > 0.$$

(ii) $$b_{r+1} = 0, \; w_{r+1} + w_{r+2} \geqq \rho_{r+1} + w_{r+2} \geqq w_{r+1},$$
$$y_{r+1} > b_{r+1}, \quad w_{r+1} > \sigma_{r+1} + w_{r+2} > 0,$$
whence $$\rho_{r+1} - \sigma_{r+1} > 0.$$

(iii) $$b_{r+1} > y_{r+1}, \; w_{r+1} > \rho_{r+1} + w_{r+2} \geqq 0,$$
$$y_{r+1} = 0, \; w_{r+1} + w_{r+2} > \sigma_{r+1} + w_{r+2} > w_{r+1},$$
whence $$\sigma_{r+1} - \rho_{r+1} > 0.$$

Hence in all cases, when $[y_r = b_r]^r_{r=2u-1}$, $y_{r+1} \neq b_{r+1}$,

$$F(Yd) \gtrless \beta \quad \text{as} \quad (-1)^r y_{r+1} \gtrless (-1)^r b_{r+1}. \tag{45}$$

II. In the second case, let β be contained in the interval $1 - w_{2s-1}$, $1 - w_{2s+1}$. Results corresponding exactly to those just proved hold in this case also.

$$F(Yd) < \beta, \quad \text{if } l = 2k, \; k < s \quad \text{or if } l = 2u - 1,$$
$$F(Yd) > \beta, \quad \text{if } l = 2k, \; k > s. \tag{46}$$

If $$l = 2s, \quad F(Yd) \gtrless \beta \quad \text{according as} \quad y_{2s} \gtrless b_{2s}. \tag{47}$$

If $[y_r = b_r]^r_{r=2s}$, $\quad y_{r+1} \neq b_{r+1}$,

$$F(Yd) \gtrless \beta \quad \text{according as} \quad (-1)^r y_{r+1} \gtrless (-1)^r b_{r+1}. \tag{48}$$

We are now in a position to solve inequalities (1).

Suppose that $$1 - \beta' = \beta'_0 + \beta'_n + (-1)^n \rho'_n, \tag{49}$$

where β'_0, β'_n, ρ'_n, b'_n, B'_n have values analogous to those of β_0, β_n, ρ_n, b_n, B in equations (29), (28), (22), (10).

Further let $\beta'' = \beta - \beta'$.

It now follows from (40) that if Y be any root of (1),

$$1 - \beta'_0 + \beta'' + \rho'_{2n+1} > F(Yd) + F(B'_{2n+1}d) > 1 - \beta'_0 + \rho'_{2n+1}.$$

That is
$$\beta'' + \rho'_{2n+1} > F(Y'd) > \rho'_{2n+1}, \tag{50}$$
where
$$Y' = Y + B'_{2n+1}.$$
Consider now the inequalities
$$\beta'' > F(Y'd) > 0. \tag{51}$$

By making n large, as many integral roots as we please of (51) can be made equal to corresponding roots of (50) and can therefore be made to exceed a corresponding root of (1) by the constant B'_{2n+1}.

The solutions of (51) have been enumerated in inequalities (41) to (48). These formulas, therefore, provide the solution of inequalities (1).

Let us now return to the example already considered, in which $d = 7/58$. We obtain on applying the method of finding the highest common factor of 7 and 58, the following equations, corresponding to equations (6) and (7).

$$\begin{array}{lll}
7 = 0 \cdot 58 + 7, & p_1 = 0 \cdot 1 + 0 = 0, & q_1 = 0 \cdot 0 + 1 = 1, \\
58 = 8 \cdot 7 + 2, & p_2 = 8 \cdot 0 + 1 = 1, & q_2 = 8 \cdot 1 + 0 = 8, \\
7 = 3 \cdot 2 + 1, & p_3 = 3 \cdot 1 + 0 = 3, & q_3 = 3 \cdot 8 + 1 = 25, \\
2 = 2 \cdot 1 + 0. & P = 2 \cdot 3 + 1 = 7. & Q = 2 \cdot 25 + 8 = 58.
\end{array}$$

That is
$$\frac{7}{58} = 0 + \frac{1}{8+}\,\frac{1}{3+}\,\frac{1}{2},$$
$$58\, w_1 = 7, \quad 58\, w_2 = 2, \quad 58\, w_3 = 1, \quad w_4 = 0.$$

We find also that
$$\frac{3}{4} = \frac{43 \cdot 5}{58} = 6\, w_1 - 0 \cdot w_2 + 2\, w_3 - \frac{\cdot 5}{58} \tag{52}$$
$$\frac{1}{4} = \frac{14 \cdot 5}{58} = 2\, w_1 - 0 \cdot w_2 + \, w_3 - \frac{\cdot 5}{58}. \tag{53}$$

Hence the roots of (3), less than Q, are to be found amongst the numbers
$$\begin{bmatrix} 6 \\ \vdots \\ 2 \end{bmatrix} q_1 + \begin{bmatrix} 2 \\ \cdot \\ 0 \end{bmatrix} q_2 + \begin{bmatrix} 2 \\ \\ 0 \end{bmatrix} q_3,$$
namely,
$$[2, 3, 4, 5, 6] + [0, 1, 2]8 + [0, 1, 2]25,$$
as may be verified by referring to the table of roots following (3). From (53) the number 2 in the first term only appears in conjunction with 0 in the

second and 1, 2 in the third term. From (52), the number 6 in the first term is omitted when it would appear in conjunction with 0 in the second term and 2 in the third. From (14) only 0 in the second term can appear along with 2 in the third. Whence the number of terms is $4 \cdot 3 \cdot 3 + 2 - 1 - 4 \cdot 2 = 29$ which agrees with the table above. But the general law of formation is more easily followed if we construct the inequalities corresponding to (51). In this case

$$\beta'' = \frac{1}{2} = \frac{29}{58} = 4w_1 - 0 \cdot w_2 + w_3.$$

From (52) $$B'_3 = 6 + 2q_3 = 56, \quad Y' = 56 + Y,$$

$$\frac{1}{2} \geqq F\left(Y' \frac{7}{58}\right) > 0.$$

The various groups are now as follows:

$$\begin{gathered} S_1 = [1,2,3,4], \ S_1' = [0,1,2,3], \\ S_2 = S_1 + [0,1,2]8, \ S_2' = S_2, 0, \\ S_3 = S_2 + [0]25, \ S_2' + [1]25, \ S_1' + [2]25, \end{gathered} \tag{54}$$

whence

$$Y - 2 \equiv Y' \equiv \left\{\begin{array}{rrrrr} & 1, & 2, & 3, & 4 \\ & 9, & 10, & 11, & 12 \\ & 17, & 18, & 19, & 20 \\ 25, & 26, & 27, & 28, & 29 \\ & 34, & 35, & 36, & 37 \\ & 42, & 43, & 44, & 45 \\ 50, & 51, & 52, & 53, & \end{array}\right\} \bmod 58,$$

which is in agreement with previous results.

The above equations (54) connecting S_1, S_2, S_3 have been written out above, because, although it would take up too much space to exhibit the laws of formation of these groups in general, still those already given are quite typical of the results obtained in the general case.

One other result which is a special case of another general theorem may be noted. The group S_1 is repeated 7 times in S_3, and therefore its average period of repetition is 58/7, which is the same thing as $1/w_1$. It is also the same thing as $1/d$, but $1/w_1$ is chosen, because the theorem states that in gen-

eral, this average period is $1/w_{2n-1}$, where b''_{2n-1} is the first coefficient in the expansion of β'' which is not zero.

In conclusion, we shall prove the theorem which determines the smallest interval between two values of Y. Let us denote this smallest interval by L.

Let
$$w_{2n-2} > \beta'' > w_{2n}. \tag{55}$$
If $\beta'' > 1 - w_1$, $L = q_1$ by (46). Otherwise, by (24), (31),
$$\beta'' = b''_{2n-1} w_{2n-1} - \rho_{2n-1}. \tag{56}$$
Also, by (41), (44) and (45)
$$F(Y'd) = y_{2n-1} w_{2n-1} - \sigma_{2n-1}, \qquad b''_{2n-1} \geqq y_{2n-1} \geqq 0, \tag{57}$$

Two values of y_{2n-1} differing by unity must lead to values of Y' differing by q_{2n-1} and in general the converse is true. For, if the expansion in (57) has coefficients satisfying inequalities (14), these inequalities will still be satisfied when y_{2n-1} is increased by one, unless either $y_{2n-1} = a_{2n}$ or $y_{2n-1} = 0$, $y_{2n} = a_{2n+1}$. The latter case is not possible when Y' is a root of (51). And hence $L = q_{2n-1}$ or q_{2n} according as 0, 1 are possible values of y_{2n-1}, or not, when the other coefficients remain unchanged.

When
$$y_{2n-1} = 0, \quad -\sigma_{2n-1} > 0.$$
If, by making σ_{2n-1} smaller, we can make it possible that $y_{2n-1} = 0, 1,$
$$b''_{2n-1} w_{2n-1} - \rho_{2n-1} > w_{2n-1} - \sigma_{2n-1}.$$
That is,
$$b''_{2n-1} > 1 \quad \text{or} \quad b''_{2n-1} = 1, \quad -\rho_{2n-1} > 0.$$

Both conditions are included if we write
$$\beta'' > w_{2n-1}.$$

Hence the theorem that, assuming inequalities (55) satisfied,
$$L = q_{2n-1} \quad \text{or} \quad q_{2n}$$
according as $\beta'' >$ or $\gtreqless w_{2n-1}$. In the example already considered
$$w_0 > 1 - w_1 > \frac{1}{2} > w_1$$
and therefore $L = q_1 = 1$.

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